The groups of automorphisms of the Witt W_n and Virasoro Lie algebras

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Abstract

Let $L_n = K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be a Laurent polynomial algebra over a field K of characteristic zero, $W_n := \mathrm{Der}_K(L_n)$, the *Witt* Lie algebra, and Vir be the *Virasoro* Lie algebra. We prove that $\mathrm{Aut}_{\mathrm{Lie}}(W_n) \simeq \mathrm{Aut}_{\mathrm{K-alg}}(L_n) \simeq \mathrm{GL}_n(\mathbb{Z}) \ltimes K^{*n}$ and $\mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vir}) \simeq \mathrm{Aut}_{\mathrm{Lie}}(W_1) \simeq \{\pm 1\} \ltimes K^*$.

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1 Introduction

In this paper, module means a left module, K is a field of characteristic zero and K^* is its group of units, and the following notation is fixed:

- $P_n := K[x_1, \ldots, x_n] = \bigoplus_{\alpha \in \mathbb{N}^n} Kx^\alpha$ is a polynomial algebra over K where $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$,
- $G_n := \mathrm{Aut}_{\mathrm{K-alg}}(P_n)$ is the group of automorphisms of the polynomial algebra P_n,
- $L_n := K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] = \bigoplus_{\alpha \in \mathbb{Z}^n} Kx^\alpha$ is a Laurent polynomial algebra,
- $\mathbb{L}_n := \mathrm{Aut}_{\mathrm{K-alg}}(L_n)$ is the group of K-algebra automorphisms of L_n,
- $\partial_1 := \frac{\partial}{\partial x_1}, \ldots, \partial_n := \frac{\partial}{\partial x_n}$ are the partial derivatives (K-linear derivations) of P_n,
- $D_n := \mathrm{Der}_K(P_n) = \bigoplus_{i=1}^n P_n \partial_i$ is the Lie algebra of K-derivations of P_n where $[\partial, \delta] := \partial\delta - \delta\partial$,
- $\mathbb{G}_n := \mathrm{Aut}_{\mathrm{Lie}}(D_n)$ is the group of automorphisms of the Lie algebra D_n,
- $W_n := \mathrm{Der}_K(L_n) = \bigoplus_{i=1}^n L_n \partial_i$ is the *Witt* Lie algebra where $[\partial, \delta] := \partial\delta - \delta\partial$,
- $\mathbb{W}_n := \mathrm{Aut}_{\mathrm{Lie}}(W_n)$ is the group of automorphisms of the Witt Lie algebra W_n,
- $\delta_1 := \mathrm{ad}(\partial_1), \ldots, \delta_n := \mathrm{ad}(\partial_n)$ are the inner derivations of the Lie algebras D_n and W_n determined by $\partial_1, \ldots, \partial_n$ where $\mathrm{ad}(a)(b) := [a, b]$,
- $\mathcal{D}_n := \bigoplus_{i=1}^n K\partial_i$,
- $\mathcal{H}_n := \bigoplus_{i=1}^n KH_i$ where $H_1 := x_1\partial_1, \ldots, H_n := x_n\partial_n$,

The group of automorphisms of the Witt Lie algebra $\mathbb{W}_n$. The aim of the paper is to find the groups of automorphisms of the Witt algebra W_n (Theorem 1.1) and the Virasoro algebra Vir (Theorem 1.2). The following lemma is an easy exercise.

- (Lemma 2.8) $\mathbb{L}_n \simeq \mathrm{GL}_n(\mathbb{Z}) \ltimes \mathbb{T}^n$ *where* $\mathrm{GL}_n(\mathbb{Z})$ *is identified with a subgroup of* $\mathbb{L}_n$ *via the group monomorphism* $\mathrm{GL}_n(\mathbb{Z}) \to \mathbb{L}_n$, $A = (a_{ij}) \mapsto \sigma_a : x_i \mapsto \prod_{j=1}^n x_j^{a_{ji}}$ *and*

$$\mathbb{T}^n := \{t_\lambda \in \mathbb{L}_n \,|\, t_\lambda(x_1) = \lambda_1 x_1, \ldots, t_\lambda(x_n) = \lambda_n x_n; \lambda \in K^{*n}\} \simeq K^{*n}$$

is the algebraic n*-dimensional torus.*

Theorem 1.1 $\mathbb{W}_n = \mathbb{L}_n$.

Structure of the proof. (i) $\mathbb{L}_n$ is a subgroup $\mathbb{W}_n$ (Lemma 2.2) via the group monomorphism

$$\mathbb{L}_n \to \mathbb{W}_n, \quad \sigma \mapsto \sigma : \partial \mapsto \sigma(\partial) := \sigma\partial\sigma^{-1}.$$

Let $\sigma \in \mathbb{W}_n$. We have to show that $\sigma \in \mathbb{L}_n$.

(ii)(crux) $\sigma(\mathcal{H}_n) = \mathcal{H}_n$ (Lemma 2.5), i.e.

$$\sigma(H) = A_\sigma H \ \text{ for some } \ A_\sigma \in \mathrm{GL}_n(K)$$

where $H := (H_1, \ldots, H_n)^T$.

(iii) $A_\sigma \in \mathrm{GL}_n(\mathbb{Z})$ (Corollary 2.7).

(iv) There exists an automorphism $\tau \in \mathbb{L}_n$ such that $\tau\sigma \in \mathrm{Fix}_{\mathbb{W}_n}(H_1, \ldots, H_n)$ (Lemma 2.10).

(v) $\mathrm{Fix}_{\mathbb{W}_n}(H_1, \ldots, H_n) = \mathbb{T}^n \subseteq \mathbb{L}_n$ (Lemma 2.12) and so $\sigma \in \mathbb{L}_n$. $\square$

The group of automorphisms of the Virasoro Lie algebra. The *Virasoro* Lie algebra $\mathrm{Vir} = W_1 \oplus Kc$ is a 1-dimensional central extension of the Witt Lie algebra W_1 where $Z(\mathrm{Vir}) = Kc$ is the centre of Vir and for all $i, j \in \mathbb{Z}$,

$$[x^i H, x^j H] = (j-i)x^{i+j}H + \delta_{i,-j}\frac{i^3 - i}{12}c \tag{1}$$

where $x = x_1$ and $H = H_1$.

Theorem 1.2 $\mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vir}) \simeq \mathbb{W}_1 \simeq \mathbb{L}_1 \simeq \mathrm{GL}_1(\mathbb{Z}) \ltimes \mathbb{T}^1$.

The key point in the proof of Theorem 1.2 is to use Theorem 1.3 of which Theorem 1.2 is a special case (where $\mathcal{G} = \mathrm{Vir}$, $W = W_1$ and $Z = Kc$, see Section 3).

Theorem 1.3 *Let $\mathcal{G}$ be a Lie algebra, Z be a subspace of the centre of $\mathcal{G}$ and $W = \mathcal{G}/Z$. Suppose that*

1. *every automorphism σ of the Lie algebra W can be extended to an automorphism $\widehat{\sigma}$ of the Lie algebra $\mathcal{G}$,*
2. *$Z \subseteq [G, G]$, and*
3. *$W = [W, W]$.*

Then, for each σ, the extension $\widehat{\sigma}$ is unique and the map $\mathrm{Aut}_{\mathrm{Lie}}(W) \to \mathrm{Aut}_{\mathrm{Lie}}(\mathcal{G})$, $\sigma \mapsto \widehat{\sigma}$, is a group isomorphism.

The groups $\mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{u}_n)$ and $\mathrm{Aut}_{\mathrm{Lie}}(D_n)$ where found in [3] and [4] respectively. The Lie algebras $\mathfrak{u}_n$ have been studied in great detail in [1] and [2]. In particular, in [1] it was proved that every monomorphism of the Lie algebra $\mathfrak{u}_n$ is an automorphism but this is not true for epimorphisms.

2 Proof of Theorem 1.1

This section can be seen as a proof of Theorem 1.1. The proof is split into several statements that reflect 'Structure of the proof of Theorem 1.1' given in the Introduction.

By the very definition, $\mathcal{H}_n = \bigoplus_{i=1}^n KH_i$ is an abelian Lie subalgebra of W_n of dimension n. Each element H of $\mathcal{H}_n$ is a unique sum $H = \sum_{i=1}^n \lambda_i H_i$ where $\lambda_i \in K$. Let us define the bilinear map

$$\mathcal{H}_n \times \mathbb{Z}^n \to K, \;\; (H, \alpha) \mapsto (H, \alpha) := \sum_{i=1}^n \lambda_i \alpha_i.$$

The Witt algebra W_n is a $\mathbb{Z}^n$-graded Lie algebra. The Witt algebra

$$W_n = \bigoplus_{\alpha \in \mathbb{Z}^n} \bigoplus_{i=1}^n K x^\alpha \partial_i = \bigoplus_{\alpha \in \mathbb{Z}^n} x^\alpha \mathcal{H}_n \tag{2}$$

is a $\mathbb{Z}^n$-graded Lie algebra, that is $[x^\alpha \mathcal{H}_n, x^\beta \mathcal{H}_n] \subseteq x^{\alpha+\beta}\mathcal{H}_n$ for all $\alpha, \beta \in \mathbb{Z}^n$. This follows from the identity

$$[x^\alpha H, x^\beta H'] = x^{\alpha+\beta}((H, \beta)H' - (H', \alpha)H). \tag{3}$$

In particular,

$$[H, x^\alpha H'] = (H, \alpha) x^\alpha H'. \tag{4}$$

So, $x^\alpha \mathcal{H}_n$ is the *weight subspace* $W_{n,\alpha} := \{w \in W_n \mid [H, w] = (H, \alpha)w\}$ of W_n with respect to the adjoint action of the abelian Lie algebra $\mathcal{H}_n$ on W_n. The direct sum (2) is the weight decomposition of W_n and $\mathbb{Z}^n$ is the *set of weights* of $\mathcal{H}_n$.

Let $\mathcal{G}$ be a Lie algebra and $\mathcal{H}$ be its Lie subalgebra. The *centralizer* $C_\mathcal{G}(\mathcal{H}) := \{x \in \mathcal{G} \mid [x, \mathcal{H}] = 0\}$ of $\mathcal{H}$ in $\mathcal{G}$ is a Lie subalgebra of $\mathcal{G}$. In particular, $Z(\mathcal{G}) := C_\mathcal{G}(\mathcal{G})$ is the *centre* of the Lie algebra $\mathcal{G}$. The *normalizer* $N_\mathcal{G}(\mathcal{H}) := \{x \in \mathcal{G} \mid [x, \mathcal{H}] \subseteq \mathcal{H}\}$ of $\mathcal{H}$ in $\mathcal{G}$ is a Lie subalgebra of $\mathcal{G}$, it is the largest Lie subalgebra of $\mathcal{G}$ that contains $\mathcal{H}$ as an ideal. Each element $a \in \mathcal{G}$ determines the derivation of the Lie algebra $\mathcal{G}$ by the rule $\mathrm{ad}(a) : \mathcal{G} \to \mathcal{G}$, $b \mapsto [a, b]$, which is called the *inner derivation* associated with a. An element $a \in \mathcal{G}$ is called a *locally finite element* if so is the inner derivation $\mathrm{ad}(a)$ of the Lie algebra $\mathcal{G}$, that is $\dim_K(\sum_{i \in \mathbb{N}} K\mathrm{ad}(a)^i(b)) < \infty$ for all $b \in \mathcal{G}$. Let $\mathrm{LF}(\mathcal{G})$ be the set of locally finite elements of $\mathcal{G}$.

The Cartan subalgebra $\mathcal{H}_n$ of W_n. A nilpotent Lie subalgebra C of a Lie algebra $\mathcal{G}$ such that $C = N_\mathcal{G}(C)$ is called a *Cartan subalgebra* of $\mathcal{G}$. We use often the following obvious observation: *An abelian Lie subalgebra that coincides with its centralizer is a maximal abelian Lie subalgebra.*

Lemma 2.1 *1. $\mathcal{H}_n = C_{W_n}(\mathcal{H}_n)$ is a maximal abelian Lie subalgebra of W_n.*

2. $\mathcal{H}_n$ is a Cartan subalgebra of W_n.

Proof. Both statements follow from (2) and (4) . $\square$

The next lemma is very useful and can be applied in many different situations. It allows one to see the group of automorphisms of a ring as a subgroup of the group of automorphisms of its Lie algebra of derivations.

Lemma 2.2 *Let R be a commutative ring such that there exists a derivation $\partial \in \mathrm{Der}(R)$ such that $r\partial \neq 0$ for all nonzero elements $r \in R$ (eg, $R = P_n, L_n$ and $\delta = \partial_1$). Then the group homomorphism*

$$\mathrm{Aut}(R) \to \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Der}(R)), \;\; \sigma \mapsto \sigma : \delta \mapsto \sigma(\delta) := \sigma\delta\sigma^{-1},$$

is a monomorphism.

Proof. If an automorphism $\sigma \in \mathrm{Aut}(R)$ belongs to the kernel of the group homomorphism $\sigma \mapsto \sigma$ then, for all $r \in R$, $r\partial = \sigma(r\partial)\sigma^{-1} = \sigma(r)\sigma\partial\sigma^{-1} = \sigma(r)\partial$, i.e. $\sigma(r) = r$ for all $r \in R$. This means that σ is the identity automorphism. Therefore, the homomorphism $\sigma \mapsto \sigma$ is a monomorphism. $\square$

The $(\mathbb{Z}, \lambda)$-grading and the filtration $\mathcal{F}_\lambda$ on W_n. Each vector $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ determines the $\mathbb{Z}$-grading on the Lie algebra W_n by the rule

$$W_n = \bigoplus_{i \in \mathbb{Z}} W_{n,i}(\lambda), \;\; W_{n,i}(\lambda) := \bigoplus_{(\lambda,\alpha)=i} x^\alpha \mathcal{H}_n, \;\; (\lambda, \alpha) := \sum_{i=1}^n \lambda_i \alpha_i,$$

$[W_{n,i}(\lambda), W_{n,j}(\lambda)] \subseteq W_{n,i+j}(\lambda)$ for all $i, j \in \mathbb{Z}$ as follows from (3) and (4). The $\mathbb{Z}$-grading above is called the $(\mathbb{Z}, \lambda)$-*grading* on W_n. Every element $a \in W_n$ is the unique sum of homogeneous elements with respect to the $(\mathbb{Z}, \lambda)$-grading on W_n,

$$a = a_{i_1} + a_{i_2} + \cdots + a_{i_s}, \;\; a_{i_\nu} \in W_{n,i_\nu}(\lambda),$$

and $i_1 < i_2 < \cdots < i_s$. The elements $l_\lambda^+(a) := a_{i_s}$ and $l_\lambda^-(a) := a_{i_1}$ are called the *leading term* and the *least term* of a respectively. So,

$$\begin{aligned} a &= l_\lambda^+(a) + \cdots, \\ a &= l_\lambda^-(a) + \cdots, \end{aligned}$$

where the three dots denote smaller and larger terms respectively. For all $a, b \in W_n$,

$$[a,b] = [l_\lambda^+(a), l_\lambda^+(b)] + \cdots, \tag{5}$$

$$[a,b] = [l_\lambda^-(a), l_\lambda^-(b)] + \cdots, \tag{6}$$

where the three dots denote smaller and larger terms respectively (the brackets on the RHS can be zero).

The Newton polygon of an element of W_n. Each element $a \in W_n$ is the unique finite sum $a = \sum_{\alpha \in \mathbb{Z}^n} \lambda_\alpha x^\alpha H_\alpha$ were $\lambda_\alpha \in K$ and $H_\alpha \in \mathcal{H}_n$. The set $\mathrm{Supp}(a) := \{\alpha \in \mathbb{Z}^n \mid \lambda_\alpha \neq 0\}$ is called the *support* of a and its convex hull in $\mathbb{R}^n$ is called the *Newton polygon* of a, denoted by $\mathrm{NP}(a)$.

Lemma 2.3 *Let a be a locally finite element of W_n. Then the elements $l_\lambda^+(a)$ and $l_\lambda^-(a)$ are locally finite for all $\lambda \in \mathbb{Z}^n$.*

Proof. The statement follows from (5) and (6). $\square$

Let $\mathrm{LF}(W_n)_h$ be the set of *homogeneous* (with respect to the $\mathbb{Z}^n$-grading on W_n) locally finite elements of the Lie algebra W_n.

Lemma 2.4 $\mathrm{LF}(W_n)_h = \mathcal{H}_n$.

Proof. $\mathcal{H}_n \subseteq \mathrm{LF}(W_n)_h$ since every element of $\mathcal{H}_n$ is a semi-simple element of W_n: for all $H = \sum_{i=1}^n \lambda_i H_i$ where $\lambda_i \in K$,

$$[H, x^\alpha H'] = (\lambda, \alpha) x^\alpha H' \;\text{ for all }\; \alpha \in \mathbb{Z}^n, \; H' \in \mathcal{H}_n. \tag{7}$$

It suffices to show that every homogeneous element $x^\alpha H'$ that does not belong to $\mathcal{H}_n$, i.e. $\alpha \neq 0$, is not locally finite. Fix i such that $\alpha_i \neq 0$. Let $\delta = \mathrm{ad}(x^\alpha H')$.

Suppose that $(H', \alpha) \neq 0$. This is the case for $n = 1$. Then

$$\delta^m (x^{2\alpha} H') = (m-1)! 2^{m-1} (H', \alpha)^m x^{(1+2m)\alpha} H' \;\text{ for }\; m \geq 1.$$

Therefore, the element $x^\alpha H'$ is not locally finite.

Suppose that $(H', \alpha) = 0$. Then necessarily $n \geq 2$. Fix $\beta \in \mathbb{Z}^n$ such that $(H', \beta) = 1$. Then

$$\delta^m(x^\beta H') = x^{\beta + m\alpha} H' \ \text{ for } \ m \geq 1.$$

Therefore, the element $x^\alpha H'$ is not locally finite. $\Box$

Lemma 2.5 *$\sigma(\mathcal{H}_n) = \mathcal{H}_n$ for all $\sigma \in \mathbb{W}_n$.*

Proof. Let $\sigma \in \mathbb{W}_n$ and $H \in \mathcal{H}_n$. We have to show that $H' := \sigma(H) \in \mathcal{H}_n$. The element H is a locally finite element, hence so is H'. By Lemma 2.3 and Lemma 2.4, the Newton polygon $\mathrm{NP}(H')$ has the single vertex 0, i.e. $H' \in \mathcal{H}_n$. $\Box$

Let $H = (H_1, \dots, H_n)^T$ where T stands for the transposition. By Lemma 2.5,

$$\sigma(H) = A_\sigma H \ \text{ for all } \ \sigma \in \mathbb{W}_n \tag{8}$$

where $A_\sigma = (a_{ij}) \in \mathrm{GL}_n(K)$ and $\sigma(H_i) = \sum_{j=1}^n a_{ij} H_j$. Let ${}^\sigma W_n$ be the W_n-module W_n twisted by the automorphism $\sigma \in \mathbb{W}_n$. As a vector space, ${}^\sigma W_n = W_n$, but the adjoint action is twisted by σ:

$$w \cdot x^\alpha H'' = [\sigma(w), x^\alpha H'']$$

for all $w \in W_n$ and $\alpha \in \mathbb{Z}^n$. The map $\sigma : W_n \to {}^\sigma W_n$, $w \mapsto \sigma(w)$, is a W_n-module isomorphism. By Lemma 2.5, every weight subspace $x^\alpha \mathcal{H}_n$ of the $\mathcal{H}_n$-module $W_n = \bigoplus_{\alpha \in \mathbb{Z}^n} x^\alpha \mathcal{H}_n$ is also a weight subspace for the $\mathcal{H}_n$-module ${}^\sigma W_n$, and vice versa. Moreover,

$$W_{n,\alpha} = x^\alpha \mathcal{H}_n = ({}^\sigma W_n)_{A_\sigma \alpha} \ \text{ for all } \ \alpha \in \mathbb{Z}^n \tag{9}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)^T \in \mathbb{Z}^n$ is a column: for all $H' = \sum_{i=1}^n \lambda_i H_i \in \mathcal{H}_n$,

$$[\sigma(H'), x^\alpha H''] = \sum_{i,j=1}^n \lambda_i a_{ij} \alpha_j x^\alpha H'' = (H', A_\sigma \alpha) x^\alpha H''. \tag{10}$$

Since $\sigma(\mathcal{H}_n) = \mathcal{H}_n$ and $\sigma : W_n \to {}^\sigma W_n$ is a W_n-module isomorphism, the automorphism σ permutes the weight components $\{W_{n,\alpha} = x^\alpha \mathcal{H}_n\}_{\alpha \in \mathbb{Z}^n}$. There is a bijection $\sigma' : \mathbb{Z}^n \to \mathbb{Z}^n$, $\alpha \mapsto \sigma'(\alpha)$, such that $\sigma(W_{n,\alpha}) = W_{n,\sigma'(\alpha)}$ for all $\alpha \in \mathbb{Z}^n$.

Lemma 2.6 *For all $\sigma \in \mathbb{W}_n$ and $\alpha \in \mathbb{Z}^n$, $\sigma'(\alpha) = A_{\sigma^{-1}} \alpha$.*

Proof. By (10),

$$\begin{aligned}(H', \sigma'(\alpha)) \sigma(x^\alpha H'') &= [H', \sigma(x^\alpha H'')] = \sigma([\sigma^{-1}(H'), x^\alpha H'']) = \sigma((H', A_{\sigma^{-1}} \alpha) x^\alpha H'') \\ &= (H'', A_{\sigma^{-1}} \alpha) \sigma(x^\alpha H'').\end{aligned}$$

Therefore, $\sigma'(\alpha) = A_{\sigma^{-1}} \alpha$. $\Box$

Corollary 2.7 *For all $\sigma \in \mathbb{W}_n$, $A_\sigma \in \mathrm{GL}_n(\mathbb{Z})$.*

Proof. This follows from Lemma 2.6. $\Box$

The group of automorphisms $\mathbb{L}_n = \mathrm{Aut}_{\mathrm{Lie}}(L_n)$. The group $\mathbb{L}_n$ contains two obvious subgroups: the *algebraic n-dimensional torus* $\mathbb{T}^n = \{t_\lambda \mid \lambda \in K^{*n}\} \simeq K^{*n}$ where $t_\lambda(x_i) = \lambda_i x_i$ for $i = 1, \dots, n$ and $\mathrm{GL}_n(\mathbb{Z})$ which can be seen as a subgroup of $\mathbb{L}_n$ via the group monomorphism

$$\mathrm{GL}_n(\mathbb{Z}) \to \mathbb{L}_n, \ \ A \mapsto \sigma_A : x_i \mapsto \prod_{j=1}^n x_j^{a_{ji}}. \tag{11}$$

For all $\alpha \in \mathbb{Z}^n$, $\sigma_A(x^\alpha) = x^{A\alpha}$. Hence $\sigma_{AB} = \sigma_A \sigma_B$ and $\sigma_A^{-1} = \sigma_{A^{-1}}$.

Lemma 2.8 $\mathbb{L}_n = \mathrm{GL}_n(\mathbb{Z}) \ltimes \mathbb{T}^n$.

Proof. The group of units L_n^* of the algebra L_n is equal to the direct product of its two subgroups $K^* \times \mathbb{X}$ where $\mathbb{X} = \{x^\alpha \,|\, \alpha \in \mathbb{Z}^n\} \simeq \mathbb{Z}^n$ via $x^\alpha \mapsto \alpha$. Since $\sigma(K^*) = K^*$ for all $\sigma \in \mathbb{L}_n$, there is a group homomorphism (where $\mathrm{Aut}_{gr}(G)$ is the group of automorphisms of a *group* G)

$$\theta : \mathbb{L}_n \to \mathrm{Aut}_{gr}(L_n/K^*), \quad \sigma \mapsto \overline{\sigma} : K^* x^\alpha \mapsto K^* \sigma(x^\alpha).$$

Notice that $\mathrm{Aut}_{gr}(L_n/K^*) \simeq \mathrm{Aut}_{gr}(\mathbb{Z}^n) \simeq \mathrm{GL}_n(\mathbb{Z})$ and $\theta|_{\mathrm{GL}_n(\mathbb{Z})} : \mathrm{GL}_n(\mathbb{Z}) \to \mathrm{Aut}_{gr}(L_n/K^*)$, $A \mapsto A$. Then $\mathbb{L}_n \simeq \mathrm{GL}_n(\mathbb{Z}) \ltimes \ker(\theta)$ but $\ker(\theta) = \mathbb{T}^n$. Clearly, $\mathbb{L}_n = \mathrm{GL}_n(\mathbb{Z}) \ltimes \mathbb{T}^n$. $\square$

Lemma 2.9 *Let $\sigma_A \in \mathbb{L}_n$ be as in (11) where $A \in \mathrm{GL}_n(\mathbb{Z})$, $\partial = (\partial_1, \ldots, \partial_n)^T$, $H = (H_1, \ldots, H_n)^T$ and $\mathrm{diag}(\lambda_{11}, \ldots, \lambda_{nn})$ be the diagonal matrix with the diagonal elements $\lambda_{11}, \ldots, \lambda_{nn}$. Then*

1. *$\sigma(\partial) = C_\sigma \partial$ where $C_\sigma = \mathrm{diag}(\sigma(x_1)^{-1}, \ldots, \sigma(x_n)^{-1}) A^{-1} \mathrm{diag}(x_1, \ldots, x_n)$.*
2. *$\sigma(H) = A^{-1} H$.*

Proof. 1. Let $\partial_i' = \sigma(\partial_i)$ and $x_j' = \sigma(x_j)$. Clearly, $\sigma(\partial) = C_\sigma \partial$ for some matrix $C_\sigma = (c_{ij}) \in M_n(L_n)$. Applying the automorphism σ to the equalities $\delta_{ij} = \partial_i * x_j$ where $i, j = 1, \ldots, n$, we obtain the equalities

$$\delta_{ij} = \sigma \partial_i \sigma^{-1} \sigma(x_j) = \partial_i' * x_j' = (\sum_{k=1}^n c_{ik} \partial_k) * \prod_{l=1}^n x_l^{a_{lj}} = (\sum_{k,l=1}^n c_{ik} x_k^{-1} a_{kj}) x_j'$$

where $i, j = 1, \ldots, n$. Equivalently, $C_\sigma \mathrm{diag}(x_1^{-1}, \ldots, x_n^{-1}) A = \mathrm{diag}(x_1'^{-1}, \ldots, x_n'^{-1})$, and statement 1 follows.

2. Statement 2 follows from statement 1:

$$\begin{aligned} \sigma(H) &= \sigma(\mathrm{diag}(x_1, \ldots, x_n)\partial) = \sigma(\mathrm{diag}(x_1, \ldots, x_n))\sigma(\partial) = \mathrm{diag}(\sigma(x_1), \ldots, \sigma(x_n)) C_\sigma \partial \\ &= \mathrm{diag}(\sigma(x_1), \ldots, \sigma(x_n))(\mathrm{diag}(\sigma(x_1)^{-1}, \ldots, \sigma(x_n)^{-1}) A^{-1} \mathrm{diag}(x_1, \ldots, x_n) = A^{-1} H. \ \square \end{aligned}$$

Let group G acts on a set S and $T \subseteq S$. Then $\mathrm{Fix}_G(T) := \{g \in G \,|\, gt = t \text{ for all } t \in T\}$ is the *fixator* of the the set T. $\mathrm{Fix}_G(T)$ is a subgroup of G.

Lemma 2.10 *Let $\sigma \in \mathbb{W}_n$. Then $\sigma(H) = A_{\sigma^{-1}} H$ for some $A_{\sigma^{-1}} \in \mathrm{GL}_n(\mathbb{Z})$ (see (8) and Lemma 2.7) and $\sigma_{A_{\sigma^{-1}}} \sigma \in \mathrm{Fix}_{\mathbb{W}_n}(H_1, \ldots, H_n)$ where $\sigma_{A_{\sigma^{-1}}} \in \mathrm{GL}_n(\mathbb{Z}) \subseteq \mathbb{L}_n$, see (11).*

Proof. The statement follows from Lemma 2.9.(2). $\square$

$$\mathrm{Sh}_n := \{s_\lambda \in G_n \,|\, s_\lambda(x_1) = x_1 + \lambda_1, \ldots, s_\lambda(x_n) = x_n + \lambda_n\}$$

is the *shift group* of automorphisms of the polynomial algebra P_n where $\lambda = (\lambda_1, \ldots, \lambda_n) \in K^n$; $\mathrm{Sh}_n \subset \mathrm{Aut}_{\mathrm{K-alg}}(P_n) \subseteq \mathrm{Aut}_{\mathrm{Lie}}(D_n)$.

Proposition 2.11 $\mathrm{Fix}_{\mathbb{W}_n}(\partial_1, \ldots, \partial_n) = \{e\}$.

Proof. Let $\sigma \in F := \mathrm{Fix}_{\mathbb{W}_n}(\partial_1, \ldots, \partial_n)$. We have to show that $\sigma = e$. Let $N := \mathrm{Nil}_{W_n}(\partial_1, \ldots, \partial_n) := \{w \in W_n \,|\, \delta_i^s(w) = 0 \text{ for some } s = s(w) \text{ and all } i = 1, \ldots, n\}$. Clearly, $N = D_n$. The automorphisms σ and σ^{-1} preserve the space $N = D_n$, that is $\sigma^{\pm 1}(D_n) \subseteq D_n$. Hence $\sigma(D_n) = D_n$ and $\sigma|_{D_n} \in \mathrm{Fix}_{G_n}(\partial_1, \ldots, \partial_n) = \mathrm{Sh}_n$, [4]. The only element s_λ of Sh_n that can be extended to an automorphism of W_n is e (since $s_\lambda(x_i^{-1}\partial_i) = (x_i + \lambda_1)^{-1}\partial_i$). Therefore, $\sigma = e$. In more detail, suppose that s_λ can be extended to an automorphism of the Witt algebra W_n and $\lambda_i \neq 0$, we seek a contradiction. Then applying s_λ to the relation $[x_i^{-1}\partial_i, x_i^2\partial_i] = 3\partial_i$ we obtain the relation $[s_\lambda(x_i^{-1}\partial_i), (x_i + \lambda_i)^2\partial_i] = 3\partial_i$. On the other hand, $[(x_i + \lambda_i)^{-1}\partial_i, (x_i + \lambda_i)^2\partial_i] = 3\partial_i$ in the Lie algebra $K(x_i)\partial_i$. Hence, $s_\lambda(x_i^{-1}\partial_i) - (x_i + \lambda_i)^{-1}\partial_i \in C := C_{K(x_i)\partial_i}((x_i + \lambda_i)^2\partial_i)$. Since $C = K \cdot (x + \lambda_i)^2\partial_i$, we see that $s_\lambda(x_i^{-1}\partial_i) \notin W_n$, a contradiction. In more detail, let $\alpha = (x_i + \lambda_i)^2$. Then $\beta\partial_i \in C$ where $\beta \in K(x_i)$ iff (where $\alpha' := \frac{d\alpha}{dx_i}$, etc) $0 = [\alpha\partial_i, \beta\partial_i] = (\alpha\beta' - \alpha'\beta)\partial_i = \alpha^2(\frac{\beta}{\alpha})'\partial_i$ iff $(\frac{\beta}{\alpha})' = 0$ iff $\frac{\beta}{\alpha} \in \ker_{K(x_i)}(\partial_i) = K$. Hence, $\beta \in K\alpha$, as required. $\square$

Lemma 2.12 $\mathrm{Fix}_{\mathbb{W}_n}(H_1,\dots,H_n)=\mathbb{T}^n$.

Proof. The inclusion $\mathbb{T}^n\subseteq F:=\mathrm{Fix}_{\mathbb{W}_n}(H_1,\dots,H_n)$ is obvious. Let $\sigma\in F$. We have to show that $\sigma\in\mathbb{T}^n$. In view of Proposition 2.11, it suffices to show that $\sigma(\partial_1)=\lambda_1\partial_1,\dots,\sigma(\partial_n)=\lambda_n\partial_n$ for some $\lambda=(\lambda_1,\dots,\lambda_n)\in K^{*n}$ since then $t_\lambda\sigma\in\mathrm{Fix}_{\mathbb{W}_n}(\partial_1,\dots,\partial_n)=\{e\}$ (Proposition 2.11), and so $\sigma=t_\lambda^{-1}\in\mathbb{T}^n$. Since $\sigma\in F$, the automorphism respects the weight components of the Lie algebra W_n, that is $\sigma(x^\alpha\mathcal{H}_n)=x^\alpha\mathcal{H}_n$ for all $\alpha\in\mathbb{Z}^n$. In particular, for $i=1,\dots,n$,

$$\partial_i'=\sigma(\partial_i)=\sigma(x_i^{-1}H_i)=x_i^{-1}\sum_{j=1}^n\lambda_{ij}H_j=-x_i^{-1}\sum_{j=1}^n\lambda_{ij}x_j\partial_j, \tag{12}$$

$\partial'=D^{-1}\Lambda D\partial$ where $D=\mathrm{diag}(x_1,\dots,x_n)$ and $D^{-1}\Lambda D\in\mathrm{GL}(L_n)$, and so $\Lambda=(\lambda_{ij})\in\mathrm{GL}_n(K)$. In view of (12), e have to show that Λ is a diagonal matrix. The elements $\partial_1,\dots,\partial_n$ commute, so do $\partial_1',\dots,\partial_n'$: for all $i,j=1,\dots,n$,

$$0=[\partial_i',\partial_j']=[x_i^{-1}\sum_{k=1}^n\lambda_{ik}H_k,x_j^{-1}\sum_{l=1}^n\lambda_{jl}H_l].$$

Therefore, foro all $i,j,l=1,\dots,n$, $\lambda_{ij}\lambda_{jl}=\lambda_{ji}\lambda_{il}$. For each $i=1,\dots,n$, let $c_i:=\sum_{j=1}^n\lambda_{ji}$. The above equalities yield the equalities

$$\sum_{j=1}^n\lambda_{ij}\lambda_{jl}=c_i\lambda_{il}\ \text{ for }\ i,l=1,\dots,n.$$

Equivalently, $\Lambda^2=\mathrm{diag}(c_1,\dots,c_n)\Lambda$. Therefore, $\Lambda=\mathrm{diag}(c_1,\dots,c_n)$ since $\Lambda\in\mathrm{GL}_n(K)$, as required. $\square$

Proof of Theorem 1.1. Let $\sigma\in\mathbb{W}_n$. We have to show that $\sigma\in\mathbb{L}_n$. By Lemma 2.10 and Lemma 2.12, $\tau\sigma\in\mathrm{Fix}_{\mathbb{W}_n}(H_1,\dots,H_n)=\mathbb{T}^n$ for some $\tau\in\mathbb{L}_n$, hence $\sigma\in\mathbb{L}_n$. $\square$

3 The group of automorphisms of the Virasoro algebra

The aim of this section is to find the group of automorphisms of the Virasoro algebra (Theorem 1.3). The key idea is to use Theorem 1.3.

Proof of Theorem 1.3. Let a K-linear map $s:W\to\mathcal{G}$ be a section to the surjection $\pi:\mathcal{G}\to W$, $a\mapsto a+Z$, i.e. $\pi s=\mathrm{id}_W$. The map σ is an injection and we identify the vector space W with its image in $\mathcal{G}$ via s. Then, $\mathcal{G}=W\oplus Z$, a direct sum of vector spaces.

(i) $\widehat{\sigma}$ *is unique*: Suppose we have another extension, say $\widehat{\sigma}_1$. Then $\tau:=\widehat{\sigma}_1^{-1}\widehat{\sigma}\in G:=\mathrm{Aut}_{\mathrm{Lie}}(\mathcal{G})$ and

$$\phi(w):=\tau(w)-w\in Z\ \text{ for all }\ w\in W,$$

where $\phi\in\mathrm{Hom}_K(W,Z)$. By condition 2, the inclusion $Z\subseteq[\mathcal{G},\mathcal{G}]=[W+Z,W+Z]=[W,W]$ implies that $\tau(z)=z$ for all $z\in Z$. For all $w_1,w_2\in W$,

$$[w_1,w_2]=[w_1,w_2]_W+z(w_1,w_2) \tag{13}$$

where $[\cdot,\cdot]$ and $[\cdot,\cdot]_W$ are the Lie brackets in $\mathcal{G}$ and W respectively and $z(w_1,w_2)\in Z$. Moreover, $[w_1,w_2]_W$ means $s([w_1,w_2]_W)$. Applying the automorphism τ to the above equality we have

$$\begin{aligned}[w_1,w_2]&=[\tau(w_1),\tau(w_2)]=\tau([w_1,w_2])=\tau([w_1,w_2]_W+z(w_1,w_2))\\&=[w_1,w_2]_W+\phi([w_1,w_2]_W)+z(w_1,w_2)=[w_1,w_2]+\phi([w_1,w_2]_W).\end{aligned}$$

Hence, $\phi([w_1,w_2]_W)=0$ for all $w_1,w_2\in W$. By condition 3, $\phi=0$, that is $\tau(w)=w$ for all $w\in W$. Together with the condition $\tau(z)=z$ for all $z\in Z$, this gives $\tau=e$. So, $\widehat{\sigma}=\widehat{\sigma}_1$.

(ii) *The map $\sigma \mapsto \widehat{\sigma}$ is a monomorphism*: Let $\widehat{\sigma}$ and $\widehat{\tau}$ be the extensions of σ and τ respectively. By the uniqueness, $\widehat{\sigma}\widehat{\tau}$ is the extension of $\sigma\tau$, that is $\widehat{\sigma\tau} = \widehat{\sigma}\widehat{\tau}$, and so the map $\sigma \mapsto \widehat{\sigma}$ is a homomorphism. Again, by the uniqueness, $\sigma \mapsto \widehat{\sigma}$ is a monomorphism.

(iii) *The map $\sigma \mapsto \widehat{\sigma}$ is an isomorphism*: By condition 1, the map $\sigma \mapsto \widehat{\sigma}$ is a surjection, hence an isomorphism, by (ii). $\square$

Proof of Theorem 1.2. The conditions of Theorem 1.3 are satisfied for the Virasoro algebra: $Z = Z(\mathrm{Vir}) = Kc$, $\mathrm{Vir}/Z \simeq W_1$, $[W_1, W_1] = W_1$ (since W_1 is a simple Lie algebra), $Z \subseteq [\mathrm{Vir}, \mathrm{Vir}]$ and each automorphism $\sigma \in \mathbb{W}_1 = \mathrm{Aut}_{\mathrm{Lie}}(W_1) = \mathrm{Aut}_{\mathrm{K-alg}}(L_1) = \mathrm{GL}_1(\mathbb{Z}) \ltimes \mathbb{T}^1$ is extended to an automorphism $\widehat{\sigma} \in \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vir})$ by the rule $\widehat{\sigma}(c) = c$. The last condition is obvious for $\sigma \in \mathbb{T}^1$ but for $e \neq \sigma \in \mathrm{GL}_1(\mathbb{Z}) = \{\pm 1\}$, i.e. $\sigma : L_1 \to L_1$, $x \mapsto x^{-1}$, i.e. $\sigma : W_1 \to W_1$, $x^i H \mapsto -x^{-1}H$ for all $i \in \mathbb{Z}$, it follows from the relations (1). $\square$

Corollary 3.1 *1. Each automorphism σ of the Witt algebra W_1 is uniquely extended to an automorphism $\widehat{\sigma}$ of the Virasoro algebra* Vir*. Moreover, $\widehat{\sigma}(c) = c$.*

2. All the automorphisms of the Virasoro algebra Vir *act trivially on its centre.*

When we drop condition 3 of Theorem 1.3, we obtain a more general result.

Corollary 3.2 *Let $\mathcal{G}$ be a Lie algebra, Z be a subspace of the centre of $\mathcal{G}$ and $W = \mathcal{G}/Z$. Fix a K-linear map $s : W \to \mathcal{G}$ which is a section to the surjection $\pi : \mathcal{G} \to W$, $a \mapsto a + Z$, and identify W with $\mathrm{im}(s)$, and so $\mathcal{G} = W \oplus Z$ (a direct sum of vector spaces). Let $\mathcal{K} := \{\tau = \tau_\phi \in \mathrm{End}_K(\mathcal{G}) \,|\, \tau(w) = w + \phi(w)$ and $\tau(z) = z$ for all $w \in W$ and $z \in Z$, $\phi \in \mathrm{Hom}_K(W, Z)$ is such that $\phi([W, W]) = 0\}$. Suppose that*

1. *every automorphism σ of the Lie algebra W can be extended to an automorphism $\widehat{\sigma}$ of the Lie algebra $\mathcal{G}$, and*
2. *$Z \subseteq [G, G]$.*

Then the short exact sequence of groups

$$1 \to \mathcal{K} \to \mathrm{Aut}_{\mathrm{Lie}}(\mathcal{G}) \stackrel{\psi}{\to} \mathrm{Aut}_{\mathrm{Lie}}(W) \to 1$$

is exact where $\psi(\sigma) : a + Z \mapsto \sigma(a) + Z$ for all $a \in \mathcal{G}$.

Proof. By condition 1, ψ is a group epimorphism. It remains to show that $\ker(\psi) = \mathcal{K}$. Let $\tau \in \ker(\psi)$. Each element $g \in \mathcal{G} = W \oplus Z$ is a unique sum $g = w + z$ where $w \in W$ and $z \in Z$. Then $\tau(w) = w + \phi(w)$ for some $\phi \in \mathrm{Hom}_K(W, Z)$. We keep the notation of the proof of Theorem 1.3. By condition 2, $Z \subseteq [\mathcal{G}, \mathcal{G}] = [W, W]$, hence $\tau(z) = z$ for all elements $z \in Z$. Applying the automorphism τ to the equality (13) yields $\phi([w_1, w_2]) = 0$ (see the proof of Theorem 1.3). It follows that $\ker(\psi) = \mathcal{K}$. $\square$

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